THE EXPERIENCE OF MATHEMATICAL BEAUTY

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ABSTRACT

Based on the statement, that the experience of mathematical beauty has a positive influence on students' motivations and attitudes towards mathematics and its study, the focus of this paper is the aesthetic component of mathematics.

First, the role of aesthetics for perception and education is addressed. The appreciation of the beauty of mathematics is one of the wellsprings of this subject, not only in research but also in school education. This should have implications for the teaching of mathematics. However the beauty making elements have not been very well analysed. In particular, it is not clear to what extent the criteria for aesthetics found in literature are in agreement with emotions of students. A study on this topic is presented below.

1. THE ROLE OF AESTHETICS FOR PERCEPTION AND EDUCATION

In the literature there are many reports concerning the use of aesthetics as a guide when formulating a scientific theory, or selecting ideas for mathematical proofs.

The first who introduced mathematical beauty as well as simplicity as criteria for a physical theory was Copernicus [3, p. 30]. Since then, these criteria have continued to play an extremely important role in developing scientific theories [3, p. 30; 4; 5]. This is especially so for truly, creative work that seems to be guided by aesthetic feeling rather than by any explicit intellectual process [12, p. 20]. Dirac, for example, tells about Schrödinger and himself [8, p. 136]:

> It was a sort of act of faith with us that any questions which describe fundamental laws of nature must have great mathematical beauty in them. It was a very profitable religion to hold and can be considered as the basis of much of our success.

Van der Waerden [18] reports that Poincaré and Hadamard pointed out the role of aesthetic feeling when choosing fruitful combinations in a mathematical solution process. More precisely, Poincaré asked how the unconscious should find out the right, that is fruitful, combinations among the many possible ones. He gave the answer: "by the sense of beauty, we prefer those combinations that we like" [18, p. 129; see also 15, p. 2047-2048].

A similar statement is given by Hermann Weyl [10, p. 209]:

> My work has always tried to unit the true with the beautiful and when I had to choose one or the other I usually chose the beautiful.

Thus theories, that have been described as extremely beautiful, as for example the general theory of relativity, have been compared to a work of art [5]; Paul Feyerabend [11] even considers science as being a certain form of art.

Mathematics and mathematical thought are obviously directed towards beauty as one profound characteristic. Papert and Poincaré [9, p. 2; 14] even believe that aesthetics play the most central role in the process of mathematical thinking. The *appreciation of mathematical beauty by students* should thus be an integral component of mathematical education [9]. But Dreyfus and Eisenberg [9] remarked in 1986, that developing an aesthetic appreciation for mathematics was not a major goal of school curricula [NCTM, 1980], and they suggested that "this is a tremendous mistake". However in the curricular guidelines of Northrhine-Westfalia, Germany [16, p. 38], the development of students' appreciation of mathematical beauty is explicitly demanded in the context of the *fostering of long-life positive mathematical views*. The importance of this demand may be stressed by the following statement given by Davis and Hersh [7, p. 169]:

> Blindness to the aesthetic element in mathematics is widespread and can account for a feeling that mathematics is dry as dust, as exciting as a telephone book, as remote as the laws of infangthief of fifteenth century Scotland. Contrariwise, appreciation of this element makes the subject live in a wonderful manner and burn as no other creation of the human mind seems to do.

In addition to the positive influence on students' attitudes towards mathematics, the experience of mathematical beauty would surely have as well *a positive influence on students' motivations for the study of mathematics*. Of course, this statement can only be confirmed on the basis of a classroom teaching that emphasizes students' aesthetic feelings.

2. CRITERIA OF AESTHETICS

If we want students to experience mathematical beauty, we first have to bring out the characteristics of mathematical aesthetics. What does it mean, for example, that a theorem, a proof, a problem, a solution of a problem (the process leading up to a solution, as well as the finished solution), a geometric figure, or a geometric construction is beautiful?

Although assessments about beauty are very personal, there is a far-reaching agreement among scholars as to what arguments are beautiful [8]. Thus it makes a sense to search for factors contributing to aesthetic appeal. Before starting on this journey, Hofstadter [14, p. 555] sounds a note of warning when suggesting, that it is impossible to define the aesthetics of a mathematical argument or structure in an inclusive or exclusive way:

> There exists no set of rules which delineates what it is that makes a peace beautiful, nor could there ever exist such a set of rules.

However we can find in the literature several indications of criteria determining the aesthetic rating.

The Pythagoreans took the view that beauty grows out of the mathematical *structure*, found in the mathematical *relationships* that bring together what are initially quite independent parts in such a way to form a unitary whole [13]. Chandrasekhar [4] names as aesthetic criteria for theories their display of "*a proper conformity of the parts to one another and to the whole*" while still showing "some *strangeness* in their proportion". Hermann Weyl [19, p. 11] states that beauty is closely connected with *symmetry*, and Ian Stewart [17, p. 91] points out that *imperfect symmetry* is often even more beautiful than exact mathematical symmetry, as our mind loves surprise. Davis and Hersh [7, p. 172] take the view that:

> A sense of strong personal aesthetic delight derives from the phenomenon that can be termed *order out of chaos*.

And they add:

> To some extent the whole object of mathematics is to create order where previously chaos seemed to reign, to extract structure and invariance from the midst of disarray and turmoil.

Allen Whitcombe [20] lists as aesthetic elements a number of vague concepts as: *structure, form, relations, visualisation, economy, simplicity, elegance, order*. Dreyfus and Eisenberg [9] state, according to a study they carried out, that *simplicity, conciseness* and *clarity* of an argument are the principle factors that contribute to the aesthetic value of mathematical thought. Further relevant aspects they name are: *structure*, *power*, *cleverness* and *surprise*. Cuoco, Goldenberg and Mark [6, p. 183] take the view that:

> The beauty of mathematics lies largely in the *interrelatedness of its ideas*. ... If students can make these connections, will they also see beauty in mathematics? We think so ...

Ebeling, Freund and Schweitzer [10, p. 230] point out, that the beautiful is as a rule connected with *complexity*; complexity is necessary, even though not sufficient, for aesthetics.

Complexity and simplicity are both named as principal factors for aesthetics: how do these notions fit together? If simplicity is named, it is mainly the simplicity of a solution of a complex problem, the simplicity of a proof to a theorem describing complex relationships, or the simplicity of representations of complex structures. It looks as if simplicity has to be combined in this way with complexity, in order to bring out aesthetic feelings [2].

The criteria for aesthetics might give us an idea of how to choose mathematical objects for presentation in classroom, if we want to bring out aesthetic feelings in the students. However, we have to consider that the criteria for aesthetics noted above, have in the main been developed by mathematicians and scientists. Hence it is not at all clear whether these criteria will point to worthwhile classroom activities which in turn will give rise to the looked for emotions of students. Furthermore, the quoted criteria for aesthetics are given by qualitative characteristics[1], and hence by their nature they are fuzzy quantities. Thus aesthetic considerations will depend on individual judgements. Thus another point of interest will be to find out whether in mathematic classes the aesthetic sensation of students can be expected to be relatively homogenous.

3. STUDENTS' JUDGEMENTS ON MATHEMATICAL BEAUTY – A STUDY

In order to gain more insight into the aesthetic feelings of students, a study was carried out by the author during 2003 in Germany.

3.1 Design of the study

The participants of the study were 108 students attending two gymnasiums. They were in grades 7 and 8 (36 students) and grades 11 and 12 (72 students). As well 9 university mathematics students were included in the study. The students were asked to work on the questionnaire given in Table 1, which had been developed by the author.

While the first two items of the questionnaire are of an open format, the third item uses a closed format to refer to a number of specific characteristics of mathematical problems that might be related to aesthetical feelings of students. In this item, beauty-making elements referred to in the literature are used:

- A number of statements refer to different nuances of simplicity (statements 2, 5, 12, 13, 15), or complexity (statements 1, 3, 7, 9, 18, 19, 27) of a mathematical problem or its solution, or to combinations of both (statements 4, 8, 16).
- The statements 24 and 25 refer to clarity and structure.
- The statement 6 refers to the characteristic of elegance.
- The statements 11 and 21 refer to the characteristic of surprise.
- The statements 17, 20 and 21 refer to symmetry and regularities.
- The statement 26 refers to the feature of power.

Based on experiences of the author as a teacher, as well as on advice given by colleagues, further possibly beauty making characteristics of a mathematical problem were included: the aspect of interest (statement 10), the feature of novelty / not

[1] Birkhoff [1] made an attempt to quantifying aesthetics in a general way, but his proposal seems not to be very convincing.

novelty (statements 13, 14), the reference to applications (statements 22, 26), the open ended problem feature (statement 23).

The students were instructed to tick as many statements in item 3 they felt were correct. Of course it might be that the statements marked by a student do not have equal weighting. However the focus of the study was to explore beauty-making elements, without emphasizing individual rankings, hence this matter was not a real issue.

Table 1: Questionnaire

What is a beautiful mathematical problem?

1. Write down one ore several mathematical problems, which appeal to you, respectively write down their contents.
2. What do you think is a “beautiful” mathematical problem?
3. Which are, in your view, characteristics of a beautiful mathematical problem?

The problem is tricky.
The problem has a simple solution.
The problem is complicated.
The problem looks complicated but it has a simple solution.
The problem is simple.
The problem has an elegant solution.
The problem is complex.
The problem and its solution are easily to be understood.
The problem is unfamiliar for me.
The topic of the problem is interesting.
The problem has a surprising solution.
The solution of the problem is obvious.
The nature of the problem is familiar to me.
The nature of the problem is new to me.
The solution of the problem can easily be guessed.
The problem looks simple but it has a complicated solution.
There are considered regular patterns or structures.
The solution of the problem is complicated.
The problem is a puzzle.
The problem is about symmetric figures.
The solution of the problem shows unexpected regularities.
The problem refers to realistic applications.
The problem has got more than one possible solution.
The problem respectively its solution are clearly structured.
The facts are presented clearly.
The solution of the problem is significant for further applications.
The problem requires a complex intellectual examination.
…

The students in grades 7 and 8 had participated at the "Math Kangaroo" competition shortly before they completed this questionnaire, and thus could name problems out of this competition when working on task 1.

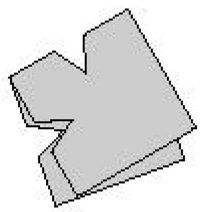
The responses given by the students of the gymnasium were differentiated according to the students' mathematical achievement in school lessons (belonging to the best third of the class/course or not).

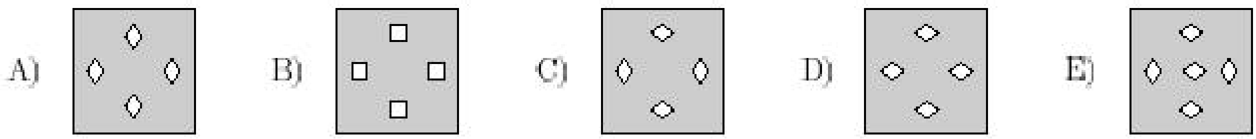
3.2 Results

Students in grades 7 and 8:

About two third of the 7th and 8th graders named *puzzles* as beautiful mathematical problems, but most of these students added that the puzzles should not be too difficult. There were no significant differences between high-achievers and low-achievers for this answer. As an example, the following problem out of the Math Kangaroo competition 2003 was named by about half of the students:

> A square piece of paper is folded twice and cut in the way you can see in the picture. How will the piece of paper look after unfolding?





Nearly all the 7th and 8th graders answered that a problem and its solution must be simple to be beautiful. As examples of this point they gave among others 1x1=1, and a problem without fractions or percents. Many students made their statement of simplicity more precise by adding that a problem is *not* beautiful if they cannot solve it by themselves, or if it is so complicated that they have no idea what to do. But a simple problem is also *not* beautiful if they had to work on this problem many times, or if it is boring, and too simple.

One of the lower-achieving students wrote in response to item 2 that a beautiful problem is a problem that can be solved by logic, and not by formulas and algorithms. A high-achieving student responding to the same item stated that a very difficult but solvable problem is beautiful.

In item 3 nearly all students ticked *simple solution / simple problem* (statements 2 and 5). About 80% marked that the *solution* of a problem should be *obvious.* Further characteristics named were an interesting topic (about 40% of all, two third of the higher achievers), a solution that can be guessed easily (about 30%), an easily understood problem / solution (about 30%), a problem that looks complicated but has a simple solution or looks simple but has a complicated solution (half of the higher-achieving students), an elegant solution (one fourth of the higher-achievers). The notion of symmetry as a beauty making element plays a subordinate role in the answers given in task 3, although symmetry is a central element of the example cited above. Table 2 shows a more detailed summary of the outcomes for item 3.

Table 2: Outcomes in item 3 of the questionnaire, grade 7 and 8[2]

Statement	*% all*	*% of low*	*% of high*
The problem has a simple solution.	100	100	100
The problem is simple.	94	100	83
The solution of the problem is obvious.	81	88	67
The topic of the problem is interesting.	42	29	67
The solution of the problem can easily be guessed.	31	42	1
The problem and its solution are easily to be understood.	31	38	17
The problem looks complicated but it has a simple solution.	22	4	58
The problem looks simple but it has a complicated solution.	17	0	100
The problem has an elegant solution.	8	0	100

The number of statements ticked by the students for item 3 differed from 3 (about half of the students) to maximum of 8. The most preferred combination was statement 2 (simple solution), statement 5 (simple problem), and statement 12 (obvious solution).

Students in grades 11 and 12:

When working on the questionnaire most of the students stated that they never really had come across a beautiful mathematical problem, but that it would be nice to do so. As "beautiful" examples out of the known problems (item 1), they mainly named very simple arithmetical problems (such as $1 \cdot a = a$), binomial formulas, and problems that can be solved by simple algorithms (e.g. systems of linear equations). Only in isolated cases (3 higher-achieving students) were some comparatively complex problems quoted (e.g. derivation of Pythagoras' theorem, calculation of volumes with integrals).

The answers to task 2 were quite varied. For the lower-achieving students a beautiful problem is mostly a problem that can be solved easily by well-known formulas or a well-known algorithm, or a problem where one sees what is to do immediately. But for these students the solution should not be too simple because that would make it

[2] There are listed only those statements ticked by at least three students.

boring, and hence it should consist of several (simple) steps. As well the solution should not be too obvious; it is better to have to first think a little about the problem. A connection to the real world is also important, as is the feeling that the problem could be useful for life.

For the higher-achieving students a beautiful problem must be a problem that he or she can solve, and it must be presented in a clear way. But these students did not stress the need for the problem to be solvable by using well-known formulas and algorithms. On the contrary, the aesthetic appeal seemed to be greater if one has to think about the problem in an unconventional way, if connections within mathematics have to be seen, if more than one well-known formula has to be used, and the successful combination of these formulas has to be found out by oneself. One student stated that a math problem is extremely beautiful if it is possible to solve it also by non-mathematical means.

Table 3: Outcomes in item 3 of the questionnaire, grades 11 and 12[3]

Statement	*% all*	*% of low*	*% of high*
The topic of the problem is interesting.	81	79	83
The problem refers to realistic applications.	65	67	63
The problem has a simple solution.	60	65	50
The nature of the problem is familiar to me.	60	71	38
The facts are presented clearly.	58	56	63
The problem is simple.	43	46	38
The problem looks complicated but it has a simple solution.	42	31	63
The problem respectively its solution are clearly structured.	39	42	33
The problem has got more than one possible solution.	36	17	75
The problem is tricky.	31	21	50
The problem is a puzzle.	28	19	46
The solution of the problem is significant for further applications.	26	27	25
The problem has a surprising solution.	22	10	46
The problem requires a complex intellectual examination.	22	8	50
The solution of the problem can easily be guessed.	22	23	21

[3] There are listed only those statements ticked by more than 15 students.

For task 3 (see Table 3) there were no great differences between high-achievers and low-achievers with regards the frequency of selection of the statements. The most important characteristic of a beautiful problem for these students is an *interesting topic* (about 80%), followed by the *reference to realistic applications* (two third of the students). Also important are a *simple problem solution* (60%), a *familiar problem nature* (60%, although more emphasized by low-achievers), and a *clear presentation* of the facts (about 60%). The characteristics of a simple problem, a complicated problem with simple solution, and a problem with a clear structure, were each marked by about 40% of the students (the second one more emphasized by high-achievers). Three fourth of the high-achievers ticked problems with more than one possible solution. In contrast to the younger students, only one third of the students (about half of the high-achievers) marked puzzles and tricky problems as beautiful. About half of the high-achievers ticked problems with a surprising solution, and problems that require a complex intellectual examination.

The number of statements ticked by an older student for item 3 was generally greater compared to the number selected by a younger student: 22% ticked 2-5 statements, 64% ticked 6-10 statements and the remaining 14% ticked 11-14 statements. Every listed statement was ticked at least once. In spite of this, there are visible favourites named by the majority of the students.

University mathematics students:

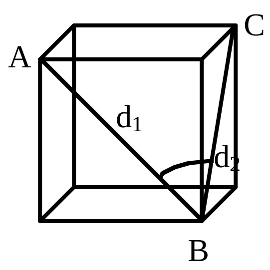
The university mathematics students all named as characteristics of a beautiful problem the *elegance of the problem solution*, as well as an *interesting topic*. *Puzzles* and *tricky* problems are also seen by most of these students as beautiful. The characteristics, *complicated looking problem with simple solution, surprising solution, more than one possible solution, reference to realistic applications* and *regular patterns or structures*, were each named by two third of the students.

The examples given are mostly complex problems or complicated looking problems with elegant, surprising, partly simple solutions. Below are the two most simple given examples:

Example 1:

Which is the angle between the diagonals d_1 and d_2 in a cube (see accompanying figure)?

Solution: Of course you might calculate with vectors, use tools of linear algebra. But, if you connect A with C you see that ABC is an equilateral triangle and you know that the angle is 60°.



Example 2:

> You have a rope that is 1m longer than the equator. Imagine, you put this rope around the equator and stretch it concentrically.
> What distance would the rope than have from the earth surface?

Solution: $\frac{1}{2p}$ m, these are round 16 cm.
This is a surprising solution, as it is contradictory to human intuition.

4. LIMITATIONS

Item 1 and 2 of the questionnaire provide qualitative statements, that may help us

- to interpret the answers given in item 3 (e.g. to get an idea of what a student means when ticking "a simple problem"),
- but also to get new ideas of what it is that makes a problem beautiful for a student.

Clearly a characteristic expressed for items 1 or 2 does not necessarily occur in the list of item 3. In this case a quantitative statement giving the weight of such a characteristic is not possible at this stage, nor was it an aim of this small project. For example, one such example is the statement that a problem can be solved by non-mathematical means (or, more exactly, by means students believe are non-mathematical). In order to explore the relevance of such an argument, further studies would have to be carried out on the pre-condition that the participating students knew already respective problems.

5. DISCUSSION

The results of the study indicate that aesthetic feelings of school students towards a mathematical problem seem to be strongly connected with interest, with the problem having realistic applications, and also giving students *feelings of security and success*: It seems to be a necessary condition that one has got to have the feeling that you could succeed in solving a certain mathematical problem, if it is going to be perceived as beautiful. In this respect it would appear that beautiful problems have to be simple enough for the group of students under observation.

But a beautiful problem is not just a simple problem. On the contrary, it has to have a certain degree of complexity: it is more beautiful if one has to think about the problem, for example if the problem is a puzzle; if the solution consists of several steps; and if one has to combine a number of formulas to get a solution. The different answers given by lower-achievers compared to those given by the higher-achievers on this point leads to the conclusion that the permissible degree of complexity for a beautiful problem depends on the mathematical ability of each individual.

With regard to university mathematics students, feelings of security and success in the solving process of a mathematical problem seems not to be a pre-condition for aesthetic feelings to arise. These students could also get pleasure from a problem which they probably could not solve by themselves.

If we want in schools to bring mathematical beauty within students' experiences, we need to use a different style of mathematical problem. We have to consider the interests of the students and choose, if possible, problems referring to realistic applications. When doing so, we have to be very aware of the abilities of our students, in order to present them problems that they can solve.

However, in my opinion it is desirable that students' aesthetic feelings are not only restricted to those problems they feel they can solve by themselves. Thus, as a further conclusion of the study, we should create phases in classrooms, which have an atmosphere that is not predominated by the demand of success. On the contrary, these phases should give the students time for leisure, time and freedom to just enjoy mathematics.

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